SUGGESTED SOLUTION TO HOMEWORK 4

JUNHAO ZHANG

Problem 1. Let $x = \sum_{k=1}^{\infty} x_k e_k$ be the expansion of a vector x with respect to a Schauder basis (e_n) in a normed space X. Show that for every $k \in \mathbb{N}$, the mapping $x \mapsto x_k$ is a linear functional on X.

Proof. Denote

$$e'_k(x) := x_k,$$

then e'_k is a mapping from X to $\mathbb{K}$. It suffices to prove that e'_k is linear. Let $\alpha, \beta \in \mathbb{K}$, $x, y \in X$, then

$$e'_k(\alpha x + \beta y) = \alpha e'_k(x) + \beta e'_k(y) = \alpha x_k + \beta y_k = e'_k(\alpha x + \beta y),$$

which implies that e'_k is linear. □

Problem 2. Show that if a normed space has n linearly independent vectors, then so does its dual space.

Proof. Let $\{x^1, \cdots, x^n\}$ be the n linearly independent vectors in the normed space X. Consider $Y = \text{span}\{x^1, \cdots, x^n\}$, then Y is a subspace of X. For each $k \in \mathbb{N}$ and $1 \le k \le n$, define the following linear functional on Y,

$$f_k(x) = x_k,$$

where $x = \sum_{i=1}^{n} x_i x^i$. We claim that f_k is a bounded linear function in Y^*.

Let us prove that there exists a real number $c > 0$ such that

$$c \sum_{i=1}^{n} |x_k| \le \|x\|,$$

for all $x \in Y$. Indeed, it is clear that the results holds for $x = 0$. For $x \neq 0$, moreover, it suffices to prove that there exists a real number c such that

$$\|x\| \ge c > 0,$$

for all $\sum_{i=1}^{n} |x_i| = 1$. Consider the set

$$S := \left\{ (x_1, \cdots, x_n) : \sum_{i=1}^{n} |x_i| = 1 \right\},$$

then S is closed and bounded in $\mathbb{R}^2$. Therefore by the Bolzano-Weierstrass theorem, S is compact. Define the following function on S,

$$f(x_1, \cdots, x_n) := \|x\|,$$

where $x = \sum_{i=1}^{n} x_i x^i$. Then f is continuous. Moreover, since S is compact, therefore f attains its minimum on the compact set S, i.e. there exists $(x'_1, \cdots x'_n) \in S$ such that

$$f(x_1, \cdots, x_n) \geq f(x'_1, \cdots, x'_n) \geq 0,$$

for all $(x_1, \cdots, x_n) \in S$. Denote $c := f(x'_1, \cdots, x'_n)$, we claim that $c > 0$. Otherwise, $c = 0$ implies that

$$\sum_{i=1}^{n} x'_i x^i = 0,$$

then since $\{x^1, \cdots, x^n\}$ are linearly independent, therefore

$$x'_i = 0,$$

for all $1 \leq i \leq n$, which contradicts to $(x'_1, \cdots, x'_n) \in S$.

Therefore for arbitrary $x \in Y$, we have

$$|f_k(x)| = |x_k| \leq \frac{1}{c}\|x\|,$$

which implies that $f_k \in Y^*$. Therefore by the Hahn-Banach theorem, there exists an extension $\tilde{f}_k \in X^*$ of f_k such that

$$\tilde{f}_k\big|_Y = f_k,$$

and

$$\|\tilde{f}_k\| = \|f_k\|.$$

We claim that $\tilde{f}_1, \cdots, \tilde{f}_n$ are linearly independent. Indeed, suppose there exists $\lambda_1, \cdots, \lambda_n$ such that

$$\sum_{i=1}^{n} \lambda_i \tilde{f}_i = 0,$$

then for each k,

$$\sum_{i=1}^{n} \lambda_i \tilde{f}_i(x^k) = 0,$$

which implies

$$\lambda_k = 0,$$

therefore $\lambda_1 = \cdots = \lambda_n$, which implies that $\tilde{f}_1, \cdots, \tilde{f}_n$ are linear independent. □

Email address: jhzhang@math.cuhk.edu.hk